

Operations on single + Multiple Random Variables - Expectations

Expected value of a Random variable:

An important concept in statistics is the mathematical expectation or simply expectation of a random variable. Consider a discrete random variable X , taking values x_1, x_2, \dots, x_n with probabilities p_1, p_2, \dots, p_n . The expectation of X is given by

$$E(X) = P_1 x_1 + P_2 x_2 + \dots + P_n x_n$$

$$= \sum_{i=1}^n x_i \cdot p_i$$

This $E(X)$ is also referred to as expected value of X or mean of X or average value of X and is also denoted as m or μ or \bar{x} .

Expected value of a random variable is its probability weighted average. If all the probabilities are same, i.e., if X takes all the values x_1, x_2, \dots, x_n with equal probability, then

$$E(X) = \frac{x_1 + x_2 + \dots + x_n}{n}$$

which is the arithmetic mean of x_1, x_2, \dots, x_n . Consider a continuous random variable X . Divide the range of X into smaller intervals, each of width Δx .

Then, the probability of X lies in the range x_i and $x_i + \Delta x$, i.e., $P(x_i \leq X \leq x_i + \Delta x)$ is

$$f(x_i) \cdot \Delta x \quad \text{i.e., } P(x_i) = f(x_i) \cdot \Delta x$$

$$\text{Now } E(X) = \sum_i x_i p(x_i) = \sum_i x_i f(x_i) \Delta x.$$

$$\text{As } \Delta x \rightarrow 0, E(X) = \int_{-\infty}^{\infty} x f_X(x) dx.$$

If a random variable y is defined as a function of x , i.e., $y = g(x)$, then

$$E(Y) = E[g(x)] = \int_{-\infty}^{\infty} g(x) f(x) dx$$

Moments of a Random variable:

For a random variable Moments of a random variable are nothing but statistical averages.

Basically, there are two sets of moments defined for a random variable.

1. Moments about origin

2. Moment about mean.

Moment about origin:-

For a random variable x n th moment about origin is denoted as $E[x^n]$.

for a discrete random variable

$$E(x^n) = \sum_i x_i^n p(x_i).$$

For a continuous random variable

$$E(x^n) = \int_{-\infty}^{\infty} x^n f(x) dx.$$

The Expectation or expected value $E(x)$ is referred to as 1st moment of the random variable about origin.

The 2nd moment about origin is $E(x^2)$ defined as $E(x^2) = \sum_i x_i^2 p(x_i)$ for a discrete random variable and $\int_{-\infty}^{\infty} x^2 f(x) dx$ for a continuous case.

Since, $E(x^2)$ is the mean of square of x , it is referred to as mean square value of (M.S value) of a random variable.

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Mean value of a random variable can be negative, but mean square value is always non-negative. i.e $E(x^2) \geq 0$.

Moments about Mean:-

These moments are also referred to as central moments.

For a random variable x , the n^{th} moment about mean is denoted as $E[(x-m)^n]$.

For a discrete random variable,

$$E[(x-m)^n] = \sum_i (x_i - m)^n p(x_i).$$

For continuous random variable,

$$E[(x-m)^n] = \int_{-\infty}^{\infty} (x-m)^n f_x(x) dx.$$

Here ' m ' is the mean of the random variable.

Variance of a Random variable:

The second central moment of a random variable x is defined as $E[(x-m)^2] = \sum_i (x_i - m)^2 p(x_i)$ for a discrete random variable and $\int_{-\infty}^{\infty} (x-m)^2 f_x(x) dx$ for continuous case.

Considering this for example

$$E[(x-m)^2] = (1-2)^2 \left(\frac{1}{2}\right) + (3-2)^2 \left(\frac{1}{2}\right) = 1.$$

Thus, the second moment about the mean gives the required variation. Hence, the second central moment of a random variable is referred to as its variance denoted by σ^2 .

The positive square root of variance is called standard deviation σ , which is a measure of the spread of a random variable about its mean.

$$\begin{aligned}
 \text{Var}(x) &= E[(x-m)^2] = \int_{-\infty}^{\infty} (x-m)^2 f_x(x) dx \\
 &= \int_{-\infty}^{\infty} (x^2 + m^2 - 2mx) f_x(x) dx \\
 &= \int_{-\infty}^{\infty} x^2 f_x(x) dx + m^2 \int_{-\infty}^{\infty} f_x(x) dx - 2m \int_{-\infty}^{\infty} xf_x(x) dx \\
 &= E(x^2) + m^2 - 2m E(x)
 \end{aligned}$$

Since $m = E(x)$

$$\text{Var}(x) = E(x^2) + [E(x)]^2 - 2[E(x)]^2.$$

$$\therefore \text{Var}(x) = E(x^2) - [E(x)]^2.$$

This result can also be verified for a discrete random variable as

$$\begin{aligned}
 \text{Var}(x) &= \sum_i (x_i - m)^2 p(x_i) = \sum_i (x_i^2 + m^2 - 2mx_i) p(x_i) \\
 &= \sum_i x_i^2 p(x_i) + m^2 \sum_i p(x_i) - 2m \sum_i x_i p(x_i) \\
 &= E(x^2) + m^2 - 2m E(x). \\
 &= E(x^2) + [E(x)]^2 - 2[E(x)]^2.
 \end{aligned}$$

$$\text{Var}(x) = E(x^2) - [E(x)]^2.$$

If mean of a random variable is zero, the second moment of the random variable i.e., variance becomes the second moment about origin i.e., mean square value. In general, central moments become moments about origin, for a random variable with zero mean.

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Problem:- Determine Mean, mean square value and variance for following exponential density functions

(a) single sided (b) double sided.

Sol:

(a) For a single sided exponential density function is given by

$$P(x) = e^{-ax}, \quad 0 \leq x \leq \infty$$

$$= 0 \quad , \text{ otherwise.}$$

$$\bar{x} = \int_{-\infty}^{\infty} x P(x) dx = \int_0^{\infty} x e^{-ax} dx,$$

$$\begin{aligned} &= x \frac{-e^{-ax}}{-a} \Big|_0^\infty - \int_0^\infty \frac{1}{-a} \cdot \frac{-e^{-ax}}{-a} dx = \frac{e^{-ax}}{-a^2} \Big|_0^\infty \\ &= -\frac{1}{a^2} (e^{-\infty} - e^0) = \frac{1}{a^2} \end{aligned}$$

mean

$$\bar{x} = \frac{1}{a^2}$$

Mean square value $\bar{x}^2 = \int_0^{\infty} x^2 p(x) dx = \int_0^{\infty} x^2 e^{-ax} dx$

$$= \frac{12}{a^2+1} = \frac{2}{a^3}.$$

$$\left[\because x^n e^{-bx} = \frac{1}{b^{n+1}} \right]$$

Variance

$$\bar{x}^2 = (\bar{x})^2 = \frac{2}{a^3} - \left(\frac{1}{a^2} \right)^2 = \frac{2a-1}{a^4}.$$

(b) For double sided exponential density function.

$$P(x) = e^{-ax}, \quad -\infty \leq x \leq 0.$$

$$= e^{-ax} \quad 0 \leq x \leq \infty$$

$$\text{Mean } \bar{x} = \int_{-\infty}^0 x e^{-ax} dx + \int_0^{\infty} x e^{-ax} dx.$$

$$= \pi \frac{e^{\alpha x}}{\alpha} \Big|_{-\infty}^0 - \int_{-\infty}^0 \frac{e^{\alpha x}}{\alpha} dx + \pi \frac{e^{-\alpha x}}{-\alpha} \Big|_0^\infty - \int_0^\infty \frac{e^{-\alpha x}}{-\alpha} dx$$

$$= -\frac{e^{\alpha x}}{\alpha^2} \Big|_{-\infty}^0 - e^{-\alpha x} \Big|_0^\infty = \left[0 - \frac{1}{\alpha^2} \right] + \frac{1}{\alpha^2} = 0.$$

Mean square $\bar{x}^2 = \int_{-\infty}^{\infty} x^2 p(x) dx = 2 \int_0^{\infty} x^2 e^{-\alpha x} dx = 2 \frac{1^2}{\alpha^2+1} = \frac{4}{\alpha^2}$.

Variance $\sigma^2 = \bar{x}^2 - (\bar{x})^2 = \bar{x}^2 = \frac{4}{\alpha^2}$.

Chebychev's Inequality:-

If the probability density function of a continuous random variable or probability mass function of a discrete random variable is known, its expected value, its variance etc. can be computed. In the reverse way, if E and $\text{var}(x)$ of a random variable x is known, the pdf of x can't be constructed. Then, if we want to compute $P\{|x - E(x)| \leq k\}$, it can't be done, just by knowing the statistical parameters of x . But, upper and lower bounds for such above probabilities can be obtained.

"If x is a random variable with mean ' m ' and variance ' σ^2 ', then for any $k > 0$,

$$P\{|x - m| \geq k\} \leq \frac{\sigma^2}{k^2}.$$

Let $f_x(x)$ be the pdf of x

$$\text{Var}(x) = \sigma^2 = \int_{-\infty}^{\infty} (x - m)^2 f_x(x) dx.$$

$$= \underbrace{\int_{-\infty}^{m-k} (x - m)^2 f_x(x) dx}_{I} + \underbrace{\int_{m-k}^{m+k} (x - m)^2 f_x(x) dx}_{II} + \underbrace{\int_{m+k}^{\infty} (x - m)^2 f_x(x) dx}_{III}.$$

If the random variable x is uniformly distributed over $[1 - \frac{1}{\sqrt{3}}, 1 + \frac{1}{\sqrt{3}}]$, compute $P\{|x-\mu| \geq \frac{3\sigma}{2}\}$ and compare it with the upper bound obtained by Chebychev's inequality.

Sol:-

$$f_x(x) = \frac{1}{\left(1 + \frac{1}{\sqrt{3}}\right) - \left(1 - \frac{1}{\sqrt{3}}\right)} = \frac{\sqrt{3}}{2}$$

For a uniform random variable over (a, b) ,

$$E(x) = \frac{b+a}{2} = \frac{\left[1 + \frac{1}{\sqrt{3}}\right] + \left[1 - \frac{1}{\sqrt{3}}\right]}{2} = 1$$

Variance $\sigma^2 = \frac{(b-a)^2}{12} = \frac{\left[\left(1 + \frac{1}{\sqrt{3}}\right) - \left(1 - \frac{1}{\sqrt{3}}\right)\right]^2}{12} = \frac{1}{9} \Rightarrow \sigma = \frac{1}{3}$

$$P\{|x-\mu| \geq \frac{3\sigma}{2}\} = P\{|x-\mu| \geq \frac{3}{2} \cdot \frac{1}{3}\} = P\{|x-\mu| \geq \frac{1}{2}\}$$

The Chebychev's bound is $P\{|x-\mu| \geq \frac{1}{2}\} \leq \frac{\sigma^2}{(\mu_2)} = \frac{4}{9}$.
Thus, upper bound for $P\{|x-\mu| \geq \frac{1}{2}\}$ is $\frac{4}{9}$.

Consider the computation of actual probability

$$\begin{aligned} P\{|x-\mu| \geq \frac{1}{2}\} &= P\{|x| \geq \frac{3}{2}\} = P\{x \leq -\frac{1}{2} \text{ and } x \geq \frac{1}{2}\} \\ &= P\{x \geq \frac{1}{2}\}. \end{aligned}$$

$$= \int_{-\frac{1}{2}}^{+\frac{1}{2}} f_x(x) dx = \frac{\sqrt{3}}{2} \left[1 + \frac{1}{\sqrt{3}} - \frac{3}{2}\right] = 0.066.$$

Let x be the outcome from rolling one die and y be the outcome from rolling a second die. Find $E[x/y]$. Since x and y are independent

$$E\left[\frac{x}{y}\right] = E[x].$$

$$E(x) = \sum x_i p(x_i) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{21}{6} = 3.5$$

Moment Generating Function! -

To find various moments of a Random Variable, different integrals are to be solved i.e. to find the mean value, $\int_{-\infty}^{\infty} x f_X(x) dx$ is to be solved. Similarly, to find mean square value, $\int_{-\infty}^{\infty} x^2 f_X(x) dx$ is to be solved and so on.

Instead of solving various integrals, moments of a random variable can be obtained by solving only one integral, named as moment generating function, (MGF) denoted by $M_X(t)$. Here, t is not time, but it is a variable.

Moment generating function of a random variable X is defined as $M_X(t) = E[e^{tX}]$.

If X is a discrete random variable,

$$M_X(t) = \sum_i e^{t x_i} p(x_i).$$

If X is a continuous Random variable,

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x) dx$$

To justify the name moment generating function, consider

$$\begin{aligned} M_X(t) &= E[e^{tX}] = E\left[1 + tX + \frac{t^2 X^2}{2!} + \frac{t^3 X^3}{3!} + \dots\right] \\ &= 1 + E(tX) + E\left[\frac{t^2 X^2}{2!}\right] + E\left[\frac{t^3 X^3}{3!}\right] + \dots \\ &= 1 + tE(X) + \frac{t^2}{2!} E(X^2) + \frac{t^3}{3!} E(X^3) + \dots \end{aligned}$$

This expansion is a polynomial in 't' whose coefficients are the moments of the random variable. Thus, moment generating function gives the moments of the random variable, but only moments unit 2, 8.

$$\text{Consider } \frac{d}{dx} M_x(t) \Big|_{t=0} = \frac{d}{dt} \left[1 + t E(x) + \frac{t^2}{2!} E(x^2) + \dots \right] \Big|_{t=0}$$

$$= E(x) = m_1 \text{ (first moment).}$$

$$\text{Similarly } \frac{d^2}{dt^2} M_x(t) \Big|_{t=0} = E(x^2) = m_2 \text{ (second moment)}$$

and so on.

Thus the n^{th} moment of a random variable can be obtained as

$$m_n = \frac{d^n}{dt^n} M_x(t) \Big|_{t=0}.$$

Problem:

A random variable x has a probability density function

$$f_x(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 2-x, & 1 \leq x < 2 \\ 0, & \text{elsewhere} \end{cases}$$

Sol: Determine the moment generating function.

The moment generating function of a density function is given by

$$\begin{aligned} M_x(t) &= \int_{-\infty}^{\infty} e^{tx} f_x(x) dx = \int_0^1 x e^{tx} dx + \int_1^2 (2-x) e^{tx} dx \\ &= \left[x \frac{e^{tx}}{t} \right]_0^1 - \int_0^1 \frac{e^{tx}}{t} dx + \int_1^2 2 e^{tx} dx - \int_1^2 x e^{tx} dx \\ &= \frac{e^t}{t} - \frac{1}{t} \left[\frac{e^t - 1}{t} \right] + 2 \left[\frac{e^{2t} - e^t}{t} \right] - \left[\frac{2e^{2t} - e^t}{t} - \frac{e^{2t} - e^t}{t^2} \right]. \end{aligned}$$

$$M_x(t) = \frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} + \frac{2e^{2t}}{t} - \frac{2e^{2t}}{t^2} + \frac{e^t}{t} + \frac{e^{2t}}{t^2} - \frac{e^t}{t^2}$$

$$= \frac{e^{2t} - 2e^t + 1}{t^2} = \frac{(e^t - 1)^2}{t^2} = M_x(t)$$

$$\begin{aligned} &\because \text{Let } u = x \\ &du = dx \\ &dv = e^{tx} dx \\ &v = \frac{e^{tx}}{t} \end{aligned}$$

Properties of Moment Generating Function

Moment Generating function of a random variable

X is defined as

$$M_X(t) = E[e^{tX}]$$

If X is a discrete random variable.

$$M_X(t) = \sum_i e^{tX_i} p(X_i)$$

If X is a continuous random variable.

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x) dx$$

1. Let X be a random variable with moment generating function $M_X(t)$. Then, the moment generating function of $Y = ax + b$ is

$$M_Y(t) = E[e^{yt}] = E[e^{(ax+b)t}]$$

$$= E[e^{atx} \cdot e^{bt}]$$

$$= e^{bt} \cdot E[e^{atx}]$$

$$\boxed{M_Y(t) = e^{bt} M_X(at)}$$

2 If $M_X(t)$ is the moment generating function of the random variable X , then, the MGF of $Y = k \cdot X$ is

$$M_Y(t) = E[e^{yt}]$$

$$= E[e^{kxt}] = E[e^{x \cdot (kt)}]$$

$$\boxed{M_Y(t) = M_X(kt)}$$

3. If $M_X(t)$ is the moment generating function of the random variable X , then, the MGF of $Y = \frac{X+a}{b}$ is

$$M_Y(t) = E[e^{yt}] = E[e^{t \cdot \frac{(X+a)}{b}}]$$

$$= E[e^{tX/b} e^{at/b}]$$

$$= e^{at/b} E[e^{X \cdot (t/b)}]$$

$$M_Y(t) = e^{at/b} \cdot M_X(t/b)$$

4. If two random variable x and y having MGF $M_X(t)$ and $M_Y(t)$ such that $M_X(t) = M_Y(t)$, then, x and y are said to have identical distribution i.e., of identical density.

Find the moment generating function of the random variable having probability density function.

$$\begin{aligned} f_X(x) &= x, 0 \leq x \leq 1 \\ &= 2-x, 1 \leq x \leq 2 \\ &= 0, \text{ elsewhere} \end{aligned}$$

$$\begin{aligned} M_X(t) &= E[e^{tx}] = \int_0^1 x e^{tx} dx + \int_1^2 (2-x) e^{tx} dx \\ \text{Consider } \int_0^1 x e^{tx} dx &= \int_0^1 x \cdot d\left(\frac{e^{tx}}{t}\right) = x \left.\frac{e^{tx}}{t}\right|_0^1 - \frac{1}{t} \int_0^1 e^{tx} dx \\ &= \frac{e^t}{t} - \frac{1}{t} \left[\frac{e^{tx}}{t}\right]_0^1 = \frac{e^t}{t} - \frac{1}{t} \left[\frac{e^t}{t} - \frac{1}{t}\right]. \end{aligned}$$

$$\begin{aligned} \text{Consider } \int_1^2 (2-x) e^{tx} dx &= \int_1^2 2 e^{tx} dx - \int_1^2 x e^{tx} dx \\ &= \frac{2e^{tx}}{t} \Big|_1^2 - \left[\frac{x e^{tx}}{t} \Big|_1^2 - \frac{1}{t} \left[\frac{e^{tx}}{t} \right]_1^2 \right] \\ &= \frac{2}{t} \left[\frac{e^{2t} - e^t}{t} \right] + \frac{1}{t^2} (3e^{2t} - 4e^t). \end{aligned}$$

Characteristic Function:-

To find the moments of a random variable, MGF is used. But, its convergence is not certain always, since it depends on ' t '.

So, an alternative to find the moments of a random variable is the characteristic function, $\mathbb{E}[e^{itX}]$.

whose convergence is made independent of the variable 't' by replacing 't' by $j\omega$ in MGF (Here ω is not frequency). The characteristic function of the random variable x is defined as

$$\phi_x(\omega) = E[e^{j\omega x}]$$

For a continuous random variable x ,

$$\phi_x(\omega) = \int_{-\infty}^{\infty} e^{j\omega x_i} f_R(x_i) dx$$

For a discrete random variable

$$\phi_x(\omega) = \sum_i e^{j\omega x_i} p(x_i).$$

Find the characteristic function of a random variable x whose pdf is

$$f_x(x) = 0 \quad \text{for } x < 0$$

$$= 1 \quad \text{for } 0 \leq x \leq 1$$

$$= 0 \quad \text{for } x > 1$$

$$\begin{aligned} \phi_x(\omega) &= E[e^{j\omega x}] = \int_{-\infty}^{\infty} e^{j\omega x} f_x(x) dx \\ &= \int_0^1 e^{j\omega x} dx = \frac{e^{j\omega x}}{j\omega} \Big|_0^1 = \frac{e^{j\omega} - 1}{j\omega}. \end{aligned}$$